

Non-rational configurations, polytopes, and surfaces

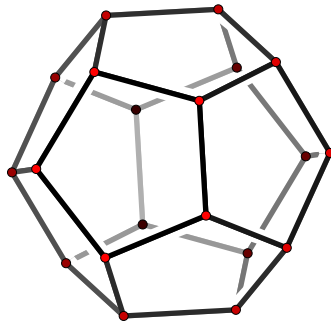
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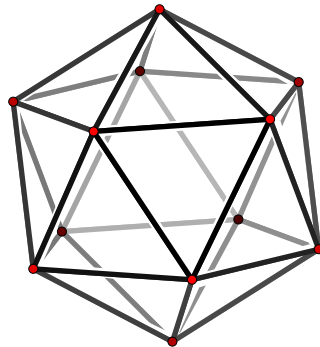
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dedicated to Micha Perles

The two most interesting “platonic solids”, the regular dodecahedron



and the regular icosahedron,



necessarily have *irrational vertex coordinates*.

Indeed, they involve regular pentagons, as faces (in the dodecahedron), or given by the five neighbors of any vertex (for the icosahedron); and a regular pentagon cannot be realized with rational coordinates, since the diagonals intersect each other in the

ratio $\tau : 1$ known as the “golden section” [7, p. 30], where $\tau = \frac{1}{2}(1 + \sqrt{5})$.

However, the dodecahedron and the icosahedron can be realized with rational coordinates if we do not require them to be precisely regular: If you perturb the vertices of a regular icosahedron “just a bit” into rational position, then taking the convex hull will yield a rational polytope that is *combinatorially equivalent* to the regular icosahedron. Similarly, by perturbing the facet planes of a regular dodecahedron a bit we obtain a dodecahedron with rational coordinates.

Indeed, *every* combinatorial type of 3-dimensional polytope can be realized with rational coordinates. For simplicial polytopes such as the icosahedron, where all faces are triangles, this can be achieved by perturbing vertex coordinates. For simple polytopes such as the dodecahedron, where all vertices have degree three, we can perturb the planes spanned by faces into rational position (that is, until the planes have equations with rational coefficients). For the case of general 3-polytopes, which may be neither simple nor simplicial, the result is not obvious, but we get it as an easy consequence of Steinitz’s proof for his (deep) theorem [29, 30] [33, Lect. 4] that every 3-connected planar graph is the graph of a convex polytope.

In view of this, it is a surprising and perhaps counter-intuitive discovery, made by Micha Perles in the sixties, that in high dimensions there are inherently *non-rational* combinatorial types of polytopes: Specifically, Perles constructed an 8-dimensional polytope with 12 vertices that can be

realized with vertex coordinates in $\mathbb{Q}[\sqrt{5}]$, but not with rational coordinates. His construction was given in terms of “Gale diagrams”, which he introduced and developed into a powerful tool for the analysis of polytopes with “few vertices”, that is, d -dimensional polytopes with $d + b$ vertices for small b .¹

Gale diagrams are a duality theory: They involve the passage to a space of complementary dimension (for a d -polytope with n vertices one arrives at an investigation in \mathbb{R}^{n-d-1}), and so the polytopes produced by Perles’ construction are hard to “visualize”. However, it was later found that non-rational polytopes may be generated from planar (non-rational) incidence configurations in a number of different ways, the simplest of which are “Lawrence extensions”. These were discovered and used, but not published, in 1980 by Jim Lawrence, then at the University of Kentucky; they first appeared in print in a paper by Billera & Munson [2] on oriented matroids. Lawrence extensions may be described via two dualization processes, but two dualizations are as good as none: and so we arrive at a “direct” construction in primal space ...

As you will see below, granted that non-rational point configurations in the plane exist (which we will see), Lawrence extensions are almost trivial to perform, and quite easy to analyze.

One might try to attribute all this to the fact that “high-dimensional geometry is weird”. However, although there is some truth to this claim, the fact that non-rational planar incidence configurations lead to non-rational geometric structures may also be seen in other instances. So, we sketch a construction by Ulrich Brehm, announced in 1997 [5] but not yet published in full, yet, which shows that there are geometric objects in \mathbb{R}^3 (namely, certain polyhedral surfaces) that are intrinsically non-rational.

Constructing instances of non-rational polytopes, or of non-rational surfaces, is not hard with the techniques we have at hand. Since the analysis and proofs become quite easy if we work with homogeneous coordinates (that is, in projective geometry), we will review this tool first; note that it is not used

in the constructions.

Much harder work — both in the careful statement of the results, and in the proofs of the theorems — is needed if one is striving for so-called universality theorems; these say that the configuration spaces of various geometric objects “are arbitrarily wild”. We will have a brief discussion later in this paper, before we end with major open problems.

Acknowledgements.

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Homogeneous coordinates and projective transformations

An abstract *configuration* is given by a set $\{p_1, \dots, p_n\}$ of n elements (“points”) and by a list which says which triples of points should be *collinear* (and that the others shouldn’t). A *realization* of the configuration is given by n points $w_1 = (x_1, y_1), \dots, w_n = (x_n, y_n) \in \mathbb{R}^2$ that satisfy the conditions, under the correspondence $p_i \leftrightarrow w_i$: The points w_i, w_j , and w_k should be collinear exactly if this had been dictated for p_i, p_j, p_k . “Being collinear” is a linear algebra condition for w_i, w_j, w_k : The points w_i, w_j , and w_k need to lie on a line, that is, be *affinely dependent*. Equivalently, the vectors $(1, x_i, y_i), (1, x_j, y_j), (1, x_k, y_k) \in \mathbb{R}^3$ need to be *linearly dependent*, that is, have determinant zero. Every realization by points w_i in \mathbb{R}^2 corresponds to a realization by vectors $v_i := (1, x_i, y_i)$ in \mathbb{R}^3 . These coordinates with a first coordinate 1 prepended are referred to as *homogeneous coordinates*.

All of what follows in this paper could in principle be discussed (and computed) in affine coordinates

¹Micha Perles, a professor of mathematics at Hebrew University in Jerusalem who just retired, is a remarkable mathematician who has published very little, but contributed a number of brilliant ideas, concepts, and proofs. His theory of Gale diagrams, as well as his construction of non-rational polytopes, were first published in the 1967 first edition of Branko Grünbaum’s book “Convex Polytopes” [10]. (See [21] or [1, Chap. 13] for another gem.)

— it would just be much more complicated.

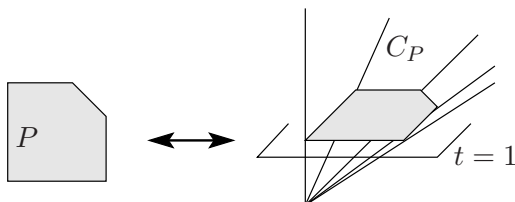
A key observation is now that linear independence is not affected if we replace any one of the vectors $v_i \in \mathbb{R}^3$ by a non-zero multiple.

Here are four fundamental facts.

- Any realization by points $w_i \in \mathbb{R}^2$ and specified affinely dependent triples yields a realization by vectors $v_i \in \mathbb{R}^3$ with specified linearly dependent triples: just pass to homogeneous coordinates.
- Conversely, any “linear” realization by vectors $v_i \in \mathbb{R}^3$ can be converted into an “affine” realization by points in \mathbb{R}^2 , by *dehomogenization*: Find a plane $at + bx + cy = 1$ that is not parallel to any one of the vectors, and rescale the vectors to lie on the plane. (That is, find a linear function $\ell(t, x, y) = at + bx + cy$ that does not vanish on any one of the vectors, and then replace v_i by $\frac{1}{\ell(v_i)}v_i$.)
- Invertible linear transformations on \mathbb{R}^3 correspond to *projective transformations* in the plane \mathbb{R}^2 .
- Any four vectors $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ such that no three of them are linearly dependent form a *projective basis*: There is a unique projective transformation that maps them to e_1, e_2, e_3 , and $e_1 + e_2 + e_3$, that is, a linear transformation that maps them to non-zero multiples of these four vectors. Indeed, if $v_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ with nonzero α_i , then consider $\{\alpha_1 v_1, \alpha_2 v_2, \alpha_3 v_3\}$ as a basis and let the linear transformation map $\alpha_i v_i$ to e_i .

Clearly, the concepts of homogenization, dehomogenization, projective transformations, and projective bases work analogously also for higher dimensions. It’s elementary real linear algebra.

For the study of convex polytopes it is also advantageous to treat them in homogenous coordinates. However, here convexity is important, and thus we have to insist on the use of *positive* rather than non-zero coefficients/multiples throughout.

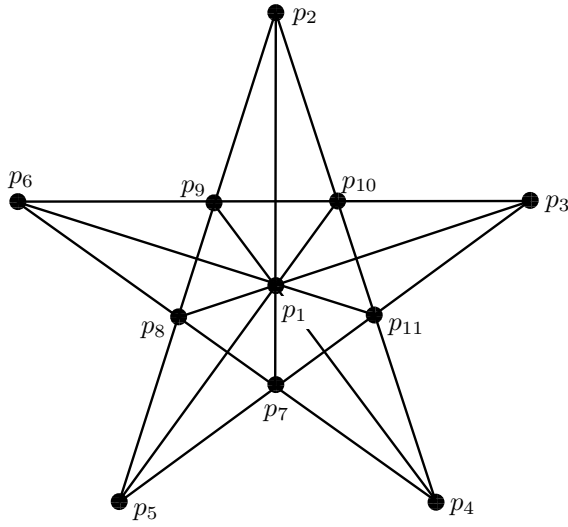


In this setting, *homogenization* is the passage from a d -dimensional convex polytope $P \subset \mathbb{R}^d$ with n vertices to a $(d + 1)$ -dimensional pointed convex polyhedral cone $C_P \subset \mathbb{R}^{d+1}$ with n extreme rays. More generally, any k -dimensional face of P corresponds to a non-empty $(k + 1)$ -dimensional face of the cone C_P , and is thus supported by a *linear* hyperplane through the origin, which is the apex of C_P . Dehomogenization allows us to pass back from any $(d + 1)$ -dimensional pointed polyhedral cone to a d -polytope. Moreover, rational d -polytopes correspond to rational $(d + 1)$ -cones, and conversely(!). (See e.g. [33, Sect. 2.6] for a more detailed discussion.)

Non-rational configurations

The insight that there are “abstract” combinatorial incidence configurations that may be geometrized with real, but not with rational coordinates is rooted deep in the history of projective geometry. (A reason for this is that addition and multiplication can be modelled by incidence configurations, via the von Staudt constructions [31, 2. Heft, 1857]; thus, polynomial equations can be encoded into point configurations. This mechanism was studied already very early in the framework of matroid theory, starting with MacLane’s fundamental 1936 paper [16], [15, pp. 147-151], where an eleven-point example was described. See Kung [15, Sect. II.1] for details and further references.) As suggested by Perles, let us look more closely at the regular pentagon.

Example 1. The *extended pentagon configuration* C_{11} is an abstract configuration on eleven points p_1, \dots, p_{11} : Its collinear triples $\{p_1, p_2, p_7\}$, $\{p_1, p_3, p_8\}$, \dots may be read off from the collinearities among the five vertices of a regular pentagon, the five intersection points of its diagonals, and the center. There are ten lines that contain more than two of these points: The five diagonals of the pentagon contain four points each, while the five lines of symmetry contain three.



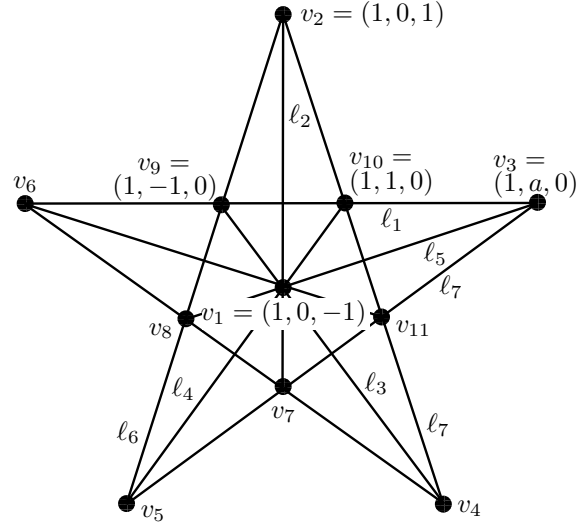
Now we want to “realize” this configuration in the rational plane, that is, find rational coordinates for all the eleven points, such that the collinearities given by the ten lines are satisfied — and such that the configuration does not “collapse”, that is, no further collinearities should occur. (We will not check that latter condition in detail, but it is important: In view of the next lemma check that there are rational coordinates for the eleven points that satisfy all ten collinearities, for example given by eleven distinct points on one line, or with the eleven points placed at the vertices of a triangle.)

Lemma 2. *The eleven-point configuration of Example 1 can be realized with coordinates in $\mathbb{Q}[\sqrt{5}]$, but not with rational coordinates.*

Proof. The calculation for this lemma is most easily done in terms of homogeneous coordinates.

In a vector realization $v_1, \dots, v_{11} \in \mathbb{R}^3$, no three of the four vectors v_1, v_2, v_9, v_{10} can be coplanar: These four vectors form a projective basis. Thus we can assume that they have, for example, the coordinates $v_1 = (1, 0, -1)$, $v_2 = (1, 0, 1)$, $v_9 = (1, -1, 0)$, and $v_{10} = (1, 1, 0)$. Furthermore, v_3 will have homogeneous coordinates $(1, a, 0)$ for some parameter $a \in \mathbb{R} \setminus \{-1, +1\}$ that we need to determine. Now it is easy (exercise!) to derive coordinates for the other vectors and equations for the lines they span, for example in the order $\ell_1 : x_2 = 0$, $\ell_2 : x_1 = 0$, $\ell_3, \ell_4, \ell_5, \ell_6$, $v_4 = (0, 1, -1)$, v_5, ℓ_7 , and then $v_7 = (1, 0, -a)$, $v_8 = (1 - a, 2a, 1 + a)$. Finally, the condition that

v_4, v_7 and v_8 need to be linearly dependent leads to the determinant equation $a^2 - 4a - 1 = 0$, that is, $a = 2 \pm \sqrt{5}$.



You should do this computation yourself. To compare results, use the labels in our figure. \square

The eleven-point “extended pentagon” example is not minimal, as you are invited to find out in the course of your computation.

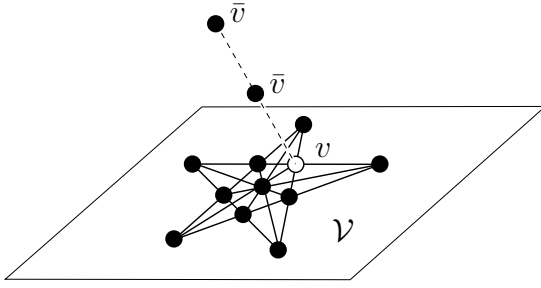
Optimization Exercise. Show that the nine-point configuration obtained from deleting the points p_6 and p_{11} also has the properties derived in Lemma 2.

Non-rational polytopes

Let \mathcal{C} be again a 2-dimensional point configuration consisting of n points, and we assume that we have a realization $\mathcal{V} = \{v_1, \dots, v_n\}$ of the configuration at hand. For the following, we should also assume that all the points v_i are distinct, that the n points do not lie on one line, and that this holds “stably”: If we delete any one of the points, then the others should not lie on a line.

If $v \in \mathcal{V}$ is any point in the configuration, a *Lawrence extension* is performed on v by replacing v by two new points \bar{v} and $\bar{\bar{v}}$ on a line through v that uses a new dimension. That is, v , \bar{v} and $\bar{\bar{v}}$ are to lie in this order on a line ℓ that intersects the affine span of \mathcal{C} only in v . Thus by this addition of two new points \bar{v} and $\bar{\bar{v}}$ and deletion of the “old” point v , the dimension of a configuration goes up by one, and so does the number of points. We will iterate this, applying Lawrence extensions

to all points in the configuration \mathcal{C} , one after the other.



Definition 3. The *Lawrence lifting* $\Lambda\mathcal{V}$ of an n -point configuration \mathcal{V} is obtained by successively applying Lawrence extensions to *all* the n points of \mathcal{V} . Thus the Lawrence lifting of a 2-dimensional n -point configuration \mathcal{V} is a $(2+n)$ -dimensional configuration that consists of $2n$ points.

Lawrence has observed that this simple construction has a number of remarkable properties. First, the order in which the Lawrence extensions are performed does not matter, since they use independent “new” directions. This may also be seen from a coordinate representation: If the n points of \mathcal{V} are given by $(x_1^i, x_2^i) \in \mathbb{R}^2$, then $\Lambda\mathcal{V}$ is given by the rows of the $2n \times (2+n)$ matrix

$$\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_n \\ \bar{\bar{v}}_1 \\ \bar{\bar{v}}_2 \\ \vdots \\ \bar{\bar{v}}_n \end{pmatrix} := \begin{pmatrix} x_1^1 & x_2^1 & 1 & & & \\ x_1^2 & x_2^2 & & 1 & & \\ \vdots & \vdots & & & \ddots & \\ x_1^n & x_2^n & & & & 1 \\ x_1^1 & x_2^1 & 2 & & & \\ x_1^2 & x_2^2 & & 2 & & \\ \vdots & \vdots & & & \ddots & \\ x_1^n & x_2^n & & & & 2 \end{pmatrix}.$$

Here \bar{v}_i and $\bar{\bar{v}}_i$ arise by lifting v_i into a new i -th direction; the specific values 1 and 2 for the “lifting heights” are not important, other positive values would give equivalent configurations.

Next, the points $\bar{v}_1, \dots, \bar{v}_n, \bar{\bar{v}}_1, \dots, \bar{\bar{v}}_n$ of $\Lambda\mathcal{V}$ are in convex position, so they are the vertices of a polytope. Moreover, for each i the pair of vertices $\bar{v}_i, \bar{\bar{v}}_i$ forms an edge of this polytope $\text{conv } \Lambda\mathcal{V}$. Indeed, it suffices to verify the last claim: From now on, let us denote the coordinates on \mathbb{R}^{2+n} by $(x_1, x_2, y_1, \dots, y_n)$. Among the points of $\Lambda\mathcal{V}$, the points \bar{v}_i and $\bar{\bar{v}}_i$ minimize the linear functional $(y_1 + \dots + y_n) - y_i$, which sums all “new variables”

except for the i th one. Thus $e_i = [\bar{v}_i, \bar{\bar{v}}_i]$ is an edge of $\Lambda\mathcal{V}$, and its endpoints are vertices.

Definition 4. The *Lawrence polytope* of the realized configuration $\mathcal{V} \subset \mathbb{R}^2$ is the convex hull of its Lawrence lifting,

$$L(\mathcal{V}) := \text{conv } \Lambda\mathcal{V} \subset \mathbb{R}^{2+n}.$$

The vertices not on e_i , i.e. the set $\Lambda\mathcal{V} \setminus \{\bar{v}_i, \bar{\bar{v}}_i\}$, form the vertex set of a facet F_i of the polytope $L(\mathcal{V})$: This is since they all minimize the linear functional y_i , and span a Lawrence polytope of dimension $1+n$.

Finally, let ℓ be any line of the original 2-dimensional configuration, which contains the points v_i ($i \in I^0$), and has the points v_j ($j \in I^-$) on one side, and the points v_k ($k \in I^+$) on the other side, for a partition $I^0 \cup I^- \cup I^+ = [n]$. Then there is a facet F^ℓ of $L(\mathcal{V})$ with vertex set $V(F^\ell) =$

$$\{\bar{v}_j : j \in I^-\} \cup \{\bar{v}_i, \bar{\bar{v}}_i : i \in I^0\} \cup \{\bar{\bar{v}}_k : k \in I^+\}.$$

To see this, let $l(x_1, x_2) = ax_1 + bx_2 + c$ be a linear function that is zero on v_i ($i \in I^0$), negative on v_j ($j \in I^-$), and positive on v_k ($k \in I^+$). From this we can easily write down a functional

$$\bar{l}(x_1, x_2, y_1, \dots, y_n) := l(x_1, x_2) + \alpha_1 y_1 + \dots + \alpha_n y_n$$

that is zero on the purported vertices of F_ℓ , and positive on all other vertices of $L(\mathcal{V})$: By just plugging in, you are led to set

$$\alpha_j := \begin{cases} 0 & \text{for } j \in I^0, \\ -l(x_1^j, x_2^j) & \text{for } j \in I^-, \\ -\frac{1}{2}l(x_1^j, x_2^j) & \text{for } j \in I^+. \end{cases}$$

Finally, we check that the face F_ℓ indeed has dimension $1+n$, so it defines a facet of $L(\mathcal{V})$.

Clearly if a configuration \mathcal{V} has rational coordinates then so do the Lawrence lifting $\Lambda\mathcal{V}$ (see the matrix above) and the Lawrence polytope $L(\mathcal{V})$. Lawrence’s remarkable observation was that the converse is true as well.

Theorem 5 (Lawrence). *Any realization of the Lawrence polytope $L(\mathcal{V})$ encodes a realization of \mathcal{V} . Thus, if $L(\mathcal{V})$ has rational coordinates, then so does \mathcal{V} .*

Proof. Let $P \subset \mathbb{R}^{2+n}$ be a polytope with the combinatorial type of $L(\mathcal{V})$. Somehow we have to start with P and “construct” \mathcal{V} from it.

For this we homogenize, and let $C_P \subset \mathbb{R}^{3+n}$ be the polyhedral cone spanned by P . Let $H_i \subset \mathbb{R}^{3+n}$ be the linear hyperplanes spanned by the n facets $F_i \subset P$ discussed above. The intersection

$$R := H_1 \cap \cdots \cap H_n$$

of these facets is a 3-dimensional linear subspace of \mathbb{R}^{3+n} : Indeed the intersection of n hyperplanes has co-dimension at most n , and the co-dimension cannot be smaller than n since for each H_i there are vertices that are not contained in H_i , but in all the other hyperplanes H_j (namely \bar{v}_i and $\bar{\bar{v}}_i$). The subspace R is the *space* where we will construct a vector representation of \mathcal{V} .

Let $E_i \subset \mathbb{R}^{3+n}$ be the 2-dimensional linear subspace that is spanned by the edge e_i (that is, by \bar{v}_i and $\bar{\bar{v}}_i$). Now e_i is contained in F_j for all $j \neq i$, but not in F_i ; thus E_i is contained in H_j for all $j \neq i$, but not in H_i . So if we intersect R with E_i , we get a linear space

$$R \cap E_i = H_i \cap E_i =: V_i$$

that is 1-dimensional. (The intersection of a 2-dimensional subspace with a hyperplane that doesn't contain it is always 1-dimensional. That's the beauty of working in vector spaces, i.e. with homogenization!) Let $v_i \in V_i \subset R$ be a non-zero vector. We claim that $v_1, \dots, v_n \in R$ give a *vector representation* of \mathcal{V} in R .

For this, consider a line ℓ of the configuration \mathcal{V} . The corresponding facet $F^\ell \subset P$ (as described above) contains the edges e^i for $i \in I^0$, but *not* the edges e^j for $j \in I^- \cup I^+$. Thus if we intersect the hyperplane $H^\ell \subset \mathbb{R}^{2+n}$ with R we get a 2-dimensional intersection that contains v_i ($i \in I^0$), but not v_j ($j \in I^- \cup I^+$) — otherwise H^ℓ would contain v_j as well as one of \bar{v}_j and $\bar{\bar{v}}_j$, but not the other one, which is impossible. This completes the proof of the claim and of the theorem. \square

Corollary 6. *The Lawrence polytope $L(\mathcal{V}_{11})$ derived from the extended pentagon configuration is a 13-dimensional non-rational polytope with 22 vertices: It can be realized with vertex coordinates in $\mathbb{Q}[\sqrt{5}]$, but not with coordinates in \mathbb{Q} .*

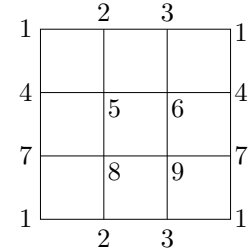
Optimization Exercise. Construct a non-rational polytope with fewer vertices, and of smaller dimension.

As a consequence of Richter-Gebert's work [23] we know that there are even 4-dimensional non-rational polytopes. Richter-Gebert's smallest example has 33 vertices.

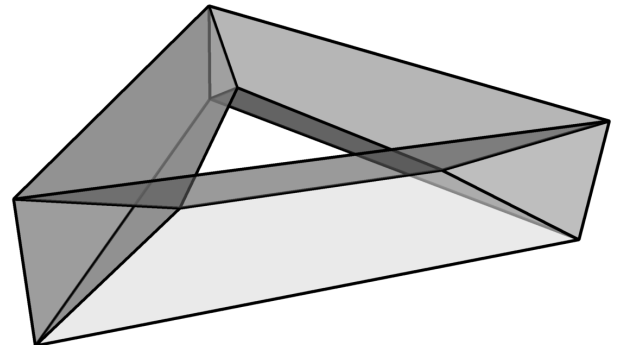
Non-rational surfaces

A *polyhedral surface* $\Sigma \subset \mathbb{R}^3$ is composed from convex polygons (triangles, quadrilaterals, etc.), which are required to intersect nicely (that is, in a common edge, a vertex, or not at all), and such that the union of all polygons is homeomorphic to a closed surface (a sphere, a torus, etc.).

The basic “gadget” that we can use to build inherently non-rational polyhedral surfaces from non-rational configurations is the “Toblerone torus” — a polyhedral nine-vertex torus built from nine quadrilateral faces. As an abstract configuration, this is the surface that you get from a 3×3 square by identifying the points on opposite edges.



You might think of such a torus as a polyhedral surface as built in 3-space from three Toblerone® (Swiss chocolate) boxes, which are long thin triangular prisms; think of the triangles at the ends as tilted (which is true for the chocolate bars, but not for their boxes).

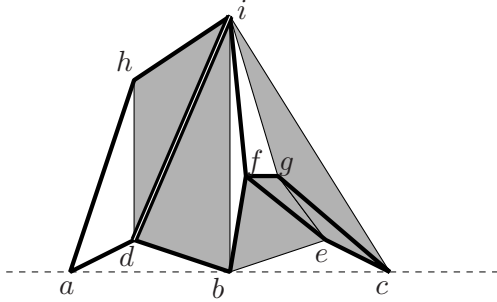


The key observation in this context is this:

Lemma 7 (Simutis [28, Thm. 6, p. 43] [9] [25]). *If you realize the toblerone torus in \mathbb{R}^3 with one quadrilateral missing, then if the eight realized quadrilaterals are flat and convex, then the missing quadrilateral is necessarily flat, and it is necessarily convex.*

The missing face of such an eight-quadrilateral Toblerone torus may be prescribed to be any given convex flat quadrilateral in 3-space: By projective transformations on 3-space, any convex flat quadrilateral can be mapped to any other one.

Now consider the following planar 9-point configuration: It consists of three black convex quadrilaterals $adih$, $bfid$, $cgfe$, and three grey shaded quadrilaterals $bdhi$, $bfge$, and $cegi$.



Think of this configuration as lying in a plane H , and using projective transformations in 3-space glue three toblerone tori with their missing faces onto the three black quadrilaterals, in such a way that the three tori all come to lie on one side of the plane H . Take three more Toblerone tori and glue them with their missing faces onto the shaded grey quadrilaterals, on the other side of H .

What you get is a partial polyhedral surface S_{48} , consisting of $6 \cdot 8 = 48$ convex quadrilaterals. It has $9 + 6 \cdot 5 = 39$ vertices, among them the nine special ones which are labelled a, b, \dots, i . It could be completed into a closed polyhedral surface by using additional triangles and quadrilaterals, but let's not do that for now.

Lemma 8 (Brehm). *In any realization of the partial surface S_{48} , the 9 special vertices a, b, \dots, i lie in a plane.*

Proof. Indeed, by Lemma 7 the six quadrilaterals are planar. It is easy to see that thus a, h, d, b, i, f

lie in one plane, and c, e, f, g, b, i lie in one plane. Both planes contain b, f, i , and since they cannot be collinear, the two planes coincide. \square

Next, says Brehm, take three copies of the partial surface S_{48} , and identify them in their copies of the vertices a_j, b_j, c_j , (for $j = 1, 2, 3$). This yields another partial surface S_{144} , consisting of $3 \cdot 48 = 144$ quadrilaterals and $3 + 3 \cdot 36 = 111$ vertices.

Lemma 9 (Brehm). *In any realization of the partial surface S_{144} , the three special vertices a, b, c lie on a line.*

Proof. Indeed, we know of three planes that the three vertices lie on. Two of these might coincide, where one 9-point configuration could lie in the upper halfplane, and one in the lower half-plane, but the third configuration then needs a different plane. Thus the three special vertices lie in the intersection of two planes. \square

From this it is quite easy to come up with, and to prove, Brehm's theorem: There are non-rational polyhedral surfaces!

Theorem 10 (Brehm 1997/2007 [5, 6]). *Glueing a copy of the partial surface S_{144} into each of the collinear triples of the 11-point pentagon configuration in \mathbb{R}^3 with flat convex quadrilaterals.*

It may be completed into a closed, embedded polyhedral surface in \mathbb{R}^3 consisting of quadrilaterals and triangles, all of whose vertex coordinates lie in $\mathbb{Q}[\sqrt{5}]$.

However, the partial surface (and hence the completed surface) does not have any rational realization.

Indeed, Lemma 9 represents already a major step on the way to Brehm's universality theorem for polyhedral surfaces.

A glimpse of universality

Following venerable traditions for example from Algebraic Geometry (where one speaks of "moduli spaces") it is natural and profitable to study not only special realizations for discrete-geometric structures such as configurations, polytopes or polyhedral surfaces, but also the *space of all correct coordinatizations*, up to affine transformations,

which is known as the *realization space* of the structure.

Why is this set a “space”, what is its structure? If we consider a planar n -point configuration \mathcal{C} , then a realization is given by an ordered set of vectors w_1, \dots, w_n , which form the rows of a matrix $W \in \mathbb{R}^{n \times 2}$. Thus a certain subset of the vector space $\mathbb{R}^{n \times 2}$ of all $2 \times n$ matrices corresponds to “correct” realizations W of “our” configuration \mathcal{C} . In all three cases (configurations, polytopes, surfaces) the set of correct realizations is a *semi-algebraic set* (more precisely, a primary semi-algebraic set defined over \mathbb{Z}): It can be described as the solution set of a finite system of polynomial equations and strict inequalities in the coordinates, with integral coefficients. For example, in the case of configurations we specify for every triple v_i, v_j, v_k that $\det(v_i, v_j, v_k)^2$ should be either zero or to be positive, which amounts to a bi-quadratic equation resp. strict inequality in the coordinates of w_i, w_j and w_k .

Any affine coordinate transformation corresponds to a column operation on the matrix $M \in \mathbb{R}^{n \times 2}$. So the realization space can be described as a quotient of the set of all realization matrices by the action of the group of affine transformations. From this point of view, it is not obvious that the realization space is a semi-algebraic set. If, however, equivalently we fix an affine basis (which in the plane means: fix the coordinates for three non-collinear points to be the vertices of a specified triangle), then this becomes clear.

Proposition 11 (see Grünbaum [10]). *The realization space of any configuration, polytope or polyhedral surface is a semi-algebraic set.*

Semi-algebraic sets can be complicated: They can

- be empty, e.g. $\{x \in \mathbb{R} : x^2 < 0\}$,
 - be disconnected, e.g. $\{x \in \mathbb{R} : x^2 > 1\}$,
 - contain only irrational points, $\{x \in \mathbb{R} : x^2 = 5\}$,
- etc. Indeed, this can easily be strengthened: Semi-algebraic sets have quite arbitrary homotopy types, singularities, or need points from large extension fields of \mathbb{Q} .

But can realization spaces for combinatorial structures be so complicated and “wild”?

It is a simple exercise to see that the realization space for a convex k -gon $P \subset \mathbb{R}^2$ has a very simple structure (equivalent to \mathbb{R}^{2k-6}). Moreover, Steinitz

[29, 30] proved in 1910 that the realization space for every 3-dimensional polytope is equivalent to \mathbb{R}^{e-6} , where e is the number of edges of P . In particular, it contains rational points. A similar result was also stated for general polytopes [24] — but it is not true.

A *universality theorem* now mandates that the realization spaces for certain combinatorial structures are as wild/complicated/interesting/strange as arbitrary semi-algebraic sets.

A blueprint is the universality theorem for oriented matroids by Nikolai Mnëv, from which he also derived a universality theorem for d -polytopes with $d + 4$ vertices:

Theorem 12 (Mnëv 1986 [17, 18]). *For every semi-algebraic set $S \subset \mathbb{R}^N$ there is for some $d > 2$ a d -polytope $P \subset \mathbb{R}^d$ with $d + 4$ vertices whose realization space $\mathcal{R}(P)$ is “stably equivalent” to S .*

Such a result of course implies that there are non-rational polytopes, that there are polytopes that have realizations that cannot be deformed into each other (counterexamples to the “isotopy conjecture”), etc. (Here we consider the realization space of the whole polytope, not only of its boundary, that is, we are considering *convex* realizations only.)

To prove such a result, a first step is to find planar configurations that encode general polynomial systems; the starting point for this are the “von Staudt constructions” [31, 2. Heft] from the 19th century, which encode addition and multiplication into incidence configurations. This produces systematically examples such as the pentagon configuration that we discussed. Then one has to show that all real polynomial systems can be brought into a suitable “standard form” (compare Shor [27]), develop a suitable concept of “stably equivalent” (compare Richter-Gebert [23]), and then go on.

In the last 20 years a number of substantial universality theorems have been obtained, each of them technical, each of them a considerable achievement. The most remarkable ones I know of today are the universality theorem for 4-dimensional polytopes by Richter-Gebert [23] (see also Günzel [11]), a universality for *simplicial* polytopes by Jaggi et al. [12], universality theorems for planar mechanical linkages by Jordan & Steiner [13] and

Kapovich & Millson [14], and the universality theorem for polyhedral surfaces by Brehm (to be published [6]).

Four problems

In the last forty years, there have been fantastic discoveries in the construction of non-rational examples, in the study of rational realizations, and in the development of universality theorems. However, great challenges remain — we take the opportunity to close here with naming four.

Small coordinates

According to Steinitz, every 3-dimensional polytope can be realized with rational, and thus also with integral vertex coordinates. However, are there *small* integral coordinates? Can every 3-polytope with n vertices be realized with coordinates in $\{0, 1, 2, \dots, p(n)\}$, for some polynomial $p(n)$? Currently, only exponential upper bounds like $p(n) \leq 533^{n^2}$ are known, due to Onn & Sturmfels [19], Richter-Gebert [23, p. 143], and finally Ribó Mor & Rote, see [22, Chap. 6].

The bipyramidal 720-cell

It may well be that non-rational polytopes occur “in nature”. A good candidate is the “first truncation” of the regular 600-cell, obtained as the convex hull of the mid points of the edges of the 600-cell, which has 600 regular octahedra and 120 icosahedra as facets. This polytope was apparently already studied by Th. Gosset in 1897; it appears with notation $\left\{ \frac{3}{3}, \frac{5}{5} \right\}$ in Coxeter [7, p. 162]. Its dual, which has 720 pentagonal bipyramids as facets, is the 4-dimensional *bipyramidal 720-cell* of Gevay [8] [20]. It is neither simple nor simplicial.

Does this polytope (equivalently: its dual) have a realization with rational coordinates?

Non-rational cubical polytopes

As argued above, it is easy to see that all types of simplicial d -dimensional polytopes can be realized with rational coordinates: “Just perturb the vertex coordinates”. For *cubical* polytopes, all of whose faces are combinatorial cubes, there is no such simple argument. Indeed, it is a long-standing open problem whether every cubical polytope has a rational realization. This is true for $d = 3$, as a special case of Steinitz’s results. But how about cubical polytopes of dimension 4? The boundary of such a polytope consists of combinatorial 3-cubes; its combinatorics is closely related with that of immersed cubical surfaces [26].

On the other hand, if we impose the condition that the cubes in the boundary have to be affine cubes — so all 2-faces are centrally symmetric — then there are easy non-rational examples, namely the *zonotopes* associated to non-rational configurations [33, Lect. 7].

Universality for simplicial 4-polytopes

There are universality theorems for simplicial d -dimensional polytopes with $d + 4$ vertices, and for 4-dimensional polytopes. But how about universality for simplicial 4-dimensional polytopes?

The realization space for such a polytope is an *open* semi-algebraic set, so it certainly contains rational points, and it cannot have singularities. One specific “small” simplicial 4-polytope with 10 vertices that has a combinatorial symmetry, but no symmetric realization, was described by Bokowski, Ewald & Kleinschmidt in 1984 [3]; according to Mnëv [17, p. 530] and Bokowski & Guedes de Oliveira [4] this example does not satisfy the isotopy conjecture, that is, the realization space is disconnected for this example. Are there 4-dimensional simplicial polytopes with more/arbitrarily complicated homotopy types?

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